Turbulence: from hydrodynamics to the solar wind plasma
-An Introduction

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Why study turbulence in solar physics?

- Coronal heating problem
- Solar wind heating problem

Matthaeus et al., 1999
What is Turbulence?
Start with hydrodynamics descriptions

• Note particularly the Navier-Stokes equation* (momentum equation)

\[
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{\nabla P}{\rho} + \nu \nabla^2 \mathbf{u}
\]

Nonlinear!

• Reynolds numbers \( R = \frac{Lu}{\nu} \)
  
  Ratio of inertial to viscous force

  Dimensionless

  *Note, usually analyzed with continuity equation \( \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \)
  and incompressible assumption \( \nabla \cdot \mathbf{u} = 0 \)
  under specified initial condition and boundary condition (B.C.).
As Reynold number increases, the symmetries permitted by the Navier-Stokes equation and boundary condition are successively broken.

Photos from Von Dyke (1982)
Two cylinders

Fully developed Turbulence: symmetries restored.

Lord Kelvin (1887): homogeneous and isotropic turbulence.

Frisch (1995)
What is Turbulence?

Turbulent or Laminar?
Inertial Forces v.s. Viscous force?
Reynold number is the essential.

Chaotic
Irregular
Mixing
Rotational, vorticity $\omega = \nabla \times u$
Dissipative

Statistical or deterministic?
What have we learned from Hydrodynamics?

Richardson eddy cascade phenomenology (1922)

Outer scale (Integrated scales)  
Anisotropic

Inertial scale (Taylor microscale)  
$\eta \ll l \ll L$

Inner scale (Kolmogorov Scales)  
Isotropic and homogeneous
What have we learned from Hydrodynamics?

Kolmogorov’s three hypotheses. At high R,

1. the small-scale turbulent motions are **statistically** isotropic (**rotation** invariant).

2. the small-scale turbulent statistics are universally and uniquely determined by \( \nu \) and energy dissipation rate \( \varepsilon \). By dimensional analysis, Kolmogorov length scale,

\[
\eta = \left( \frac{\nu^3}{\varepsilon} \right)^{1/4}
\]

3. the inertial range \((\eta<<l<<L)\) is **homogeneous** (**translation** invariant). Statistics here are universally and uniquely determined by the scale \( l \) \((1/k)\) and energy dissipation rate \( \varepsilon \), independent of \( \nu \). By dimensional analysis,

\[
E(k) \sim \varepsilon^{2/3} k^{-5/3}
\]

Kolmogorov (1941), K41
K41 dimensional analysis

\[ \frac{1}{2} \langle u^2 \rangle = \int_0^\infty E(k) \, dk \]

Thus, \( E(k) \) has dimension \( L^3/T^2 \)

Dimension of \( \varepsilon \) (energy dissipation rate per unit mass) is \( L^2/T^3 \)

K41 assumes \( E(k) \) only depends on \( \varepsilon \) and \( k \),

Then, we must have

\[ E(k) \sim \varepsilon^{2/3} k^{-5/3} \]
One example of the experimental success of K41

\[ E(k) \sim \varepsilon^{2/3} k^{-5/3} \]

Champagne, 1978
Self-similar

1-D example: brownian motion

The “general aspect” (statistical properties) within the magnification window is independent of where the window is positioned!

Frisch, 1995
Self-similar: preservation of structure function

- Self-Similar (symmetries: isotropic and homogeneous) in the inertial range (equivalent of the universal assumption in K41). There exists a scaling exponent $h$ for the $1^{\text{st}}$ order structure function $\delta u(l)$ such that

$$\delta u(\lambda l) = \lambda^h \delta u(l)$$

where increment $\delta u_i(l) = u_i(x) - u_i(x - l)$

The $p$-th order structure function thus

$$S_p(l) = \langle (\delta u_i(l))^p \rangle \propto l^{\zeta_p}$$

where $\zeta_p = p/h$.

K41-3 states that $S_p$ only depends on $\varepsilon$ and $l$. By dimensional analysis, the second order structure function $S_2 \sim \varepsilon^{2/3} l^{2/3}$. Therefore $h=1/3$ and $\zeta_p = p/3$. And in fact $S_p(l) \sim \varepsilon^{p/3} l^{p/3}$. 
Intermittency: dissipation range (high $k$) is not self-similar! (Batchelor and Townsend, 1949)

Example

- Velocity signal from a jet with $R=700$

Heating is bursty, patchy, and non-uniform

- Same signal subject to high-pass filtering, showing intermittent bursts

Gagne 1980
More examples of intermittent functions

Reality is seldom so “black and white”. How intermittent? Need to introduce intermittency measurement.
Measure of intermittency
Quantitatively: Kurtosis (Flatness)
Visually: PDF ($\delta u_i$)* deviation from Gaussian

Fourth moment around mean divided by the fourth power of standard deviation

$$k(l) = \frac{\mu_4}{\sigma^4} = \frac{\left\langle (\delta u_i(l) - \langle \delta u_i(l) \rangle)^4 \right\rangle}{\left\langle (\delta u_i(l) - \langle \delta u_i(l) \rangle)^2 \right\rangle^2} = \frac{\langle (\delta u_i(l))^4 \rangle}{\langle (\delta u_i(l))^2 \rangle^2}$$

again, velocity increment

$$\delta u_i(l) = u_i(x) - u_i(x - l)$$

Note $k$ is scale ($l$) dependent
The larger the kurtosis, the more intermittent

Perfectly self-similar case:
Gaussian signals (Normal function). Gaussian fluctuations have a flatness of 3, independent of filtering frequency.

*PDF=probability distribution function

Subedi et al., 2014
Intermittency visualization: vorticity filaments

Vorticity field (Vincent and Meneguzzi, 1991)

Vorticity filament (high concentration of vorticity) in turbulent water (Boon et al., 1993)
K41, data, models that modifies K41

Frisch, 1995

Black circles, white squares and black triangles are data from Anselmet, (1984)
von Karman decay in Hydrodynamics (3rd order law)

**Similarity decay** was suggested by Taylor (1935) and made precised by von Karman and Howarth (1938). It postulates the preservation of shape of 2 point correlation functions during Decay (Essentially, a rephrasing of K41).

Derive $\varepsilon = -aU^3/L$

and $dL/dt = bU$

where $a$ and $b$ are constants.

Decay (dissipation) rate $\varepsilon$ is controlled by $U = \langle u^2 \rangle^{1/2}$ (amplitude) and $L$ in the outer scale, independent of viscosity (or detail dissipation mechanism)!

Experimentally validated in wind tunnel measurements (Batchelor and Townsend, 1949)

Energy decay rate $\varepsilon$ is written as $dU^2/dt$

In the plot
From a fluid perspective, how does turbulence in the solar wind plasma differ from hydrodynamic Turbulence?

Navier-stocks equation becomes

\[
\frac{\partial u}{\partial t} + u \cdot \nabla u = -\frac{\nabla P}{\rho} + \nu \nabla^2 u + B \cdot \nabla B
\]

Note, here, B is written in Alfvén unit (same unit as velocity u)

Additional variable B
Additional nonlinear term
Need one more equation
From a fluid perspective, how does turbulence in the solar wind plasma differ from hydrodynamic Turbulence?

- Maxwell’s Equations
  \[
  \nabla \cdot \mathbf{E} = \rho \\
  \nabla \cdot \mathbf{B} = 0 \\
  \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} + \mathbf{J} \\
  \nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J}
  \]

- Ohm’s law \( \mathbf{J} = \sigma (\mathbf{E} + \mathbf{u} \times \mathbf{B}) \)

- Magnetic Reynold number \( R_m = R = Lu/\eta \)

\( \eta = 1/\sigma \) is the magnetic diffusivity

Eliminate \( \mathbf{E} \), \( \rightarrow \) Induction equation

\[
\frac{\partial \mathbf{B}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{u} + \eta \nabla^2 \mathbf{B}
\]

Nonlinear! Nonlinear!

Again, we can write \( \mathbf{B} \) in Alfvén unit in the induction equation and thereafter.
Magneto-hydrodynamic (MHD) Turbulence

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = -\frac{\nabla P}{\rho} + \nu \nabla^2 u + B \cdot \nabla B$$

$$\frac{\partial B}{\partial t} + u \cdot \nabla B = B \cdot \nabla u + \eta \nabla^2 B$$

$B$ can be split into a mean field $B_0$ and a fluctuating field $b$, $B = B_0 + b$. Define new variables to replace $b$ and $u$, the Elsässer (1950) variables $z^\pm = u \pm b$.

We can rewrite our equations into

$$\frac{\partial z^\pm}{\partial t} + (B_0 \cdot \nabla)z^\pm + (z^\mp \cdot \nabla)z^\pm = -\nabla P + \nu_+ \nabla^2 z^\pm + \nu_- \nabla^2 z^\mp$$

Where $\nu_\pm = \frac{1}{2}(\nu \pm \eta)$,

Nonlinear interactions occur between the $z^\pm$. 
Outer scales $L^\pm$ : e-folding definition

Two-point correlation

\[ R^\pm(l) = \frac{\langle z^\pm(x) \cdot z^\pm(x+l) \rangle}{\sigma_{z^\pm}^2} \]

Find $L^\pm$ such that $R^\pm(L^\pm) = 1/e$
Relevant time scales and turbulent models

- Alfven time $\tau_A = L^\pm/(kB_0)$
- Nonlinear time $\tau_{NL}^\pm = L^\pm/(kz_k^\pm)$

$$\frac{\partial z^\pm}{\partial t} \mp (B_0 \cdot \nabla)z^\pm + (z^\mp \cdot \nabla)z^\pm = -\nabla P + \nu_+ \nabla^2 z^\pm + \nu_- \nabla^2 z^\mp$$

- IK: Iroshnikov (1964) and Kraichnan (1965) assumed $z^+$ and $z^-$ interact weakly and linearized the equations with $\tau_A$ being the relevant time, they derived $E^u(k) \sim E^b(k) \sim (\varepsilon B_0)^{1/2} k^{-3/2}$.
- K41-like: Marsch (1990) assumed fundamentally nonlinear. $\tau_{NL}^\pm$ is the interaction time for eddies, they derived $E^\pm(k) \sim (\varepsilon^\pm)^{4/3} (\varepsilon^-)^{-2/3} k^{-5/3}$.

- Critical balance (Goldreich and Scridhar, 1995): “compound” version of IK and K41-like descriptions: $\tau_A \sim \tau_{NL}$. They derived $E_\perp(k_\perp) \propto k_\perp^{-5/3}$ and $E_\parallel(k_\parallel) \propto k_\parallel^{-2}$.
Spectra: solar wind observations

Magnetic field energy spectrum (Matthaeus and Goldstein, 1982)

Velocity spectrum (Podesta et al., 2007)
Complication: $B_0$

- In the strong $B_0$ case ($B_0 \gg b$), the turbulent spectrum splits into two parts: an essentially **2D Turbulence** spectrum with both $u$ and $b$ perpendicular to $B_0$, and a weaker and more nearly **isotropic spectrum of Alfven waves** (Montgomery and Turner, 1981, Montgomery and Matthaeus, 1995).

- MHD simulations: Mean Magnetic field $B_0$ suppresses the energy cascade along the direction of the mean magnetic field $\rightarrow$ **anisotropy** (Shebalin et al. 1983).
2D Turbulence v.s. Slab

Bruno and Carbone (2013) Review:

• Helio (0.3-1AU) data in the slow wind, Interplanetary solar wind, 74-95% 2D turbulence and 5-26% slab (Bieber et al., 1996).

• In the polar wind, 50% 2D turbulence and 50% of slab (Smith, 2003).

• Dasso et al. (2005), using 5 years of spacecraft observations at roughly 1 AU, showed that fast streams are dominated by fluctuations with wavevectors quasi-parallel to the local magnetic field (slab), while slow streams are dominated by quasi-perpendicular fluctuation wavevectors (2D turbulence).
MHD perspective: Cross helicity $H_c = \int u \cdot BdV$

High alignment of $\mathbf{b}$ and $\mathbf{u}$ (corresponds to maximum $H_c$) results $z^+$ and $z^-$ alignment and thus reduces the nonlinear term in the MHD equation. Wave and linear terms may dominate. On the contrary, low $H_c$ corresponds to a more nonlinearly turbulent plasma.

$$\frac{\partial z^\pm}{\partial t} \mp (B_0 \cdot \nabla)z^\pm + (z^\mp \cdot \nabla)z^\pm = -\nabla P + \nu_+ \nabla^2 z^\pm + \nu_- \nabla^2 z^\mp$$

Observationally, Roberts et al. (1987a, 1987b) find that when $H_c$ is nearly maximal in fast wind from 0.3-1AU, there was little evidence of turbulent evolution. Instead, fluctuations are highly Alfvénic. On the other hand, Matthaeus and Goldstein (1982) find that for (stationary) intervals spanning several days, the spectrum of B is very close to k41’s -5/3 scaling.
Alfven ratio $r_A = E_u / E_b$

Special case: $u=b$ and $u$ alignment with $b$ ($r_A=1$ and maximum $H_x$)

$$\frac{\partial z^\pm}{\partial t} \mp (B_0 \cdot \nabla) z^\pm + (z^\mp \cdot \nabla) z^\pm = -\nabla P + \nu_+ \nabla^2 z^\pm + \nu_- \nabla^2 z^\mp$$

$z^-$ vanishes

Left with $z^+=2b=2u$ and a simpler equation that is linearizable

$$\frac{\partial z^+}{\partial t} - (B_0 \cdot \nabla) z^+ = -\nabla P + \nu_+ \nabla^2 z^+$$

Fluctuations can be highly Alfvenic.

Caution: Special case does not represent solar wind general condition!
Turbulence in solar wind is dynamically active, not just a remnant of turbulence in the corona.
Solar wind spectral break at high $k(s)$: Dissipation

Goldstein et al., 2015
Dissipation

1. Intermittent dissipation by non-linear coherent structures (Matthaeus and Montgomery, 1980): primarily current sheets (and related reconnection). Heating is bursty, patchy, and non-uniform.

2. Resonant damping of Incoherent Waves
   - Landau damping of Kinetic Alfven Wave (e.g., Chandran et al., 2010, Howes et al., 2011)
   - Whistler (e.g., Chang et al., 2011)

Or both 1 and 2? And Each dominates at different conditions?
Dissipation: requires kinetic descriptions

• MHD is not adequate to address dissipation
• Need investigations that can resolve the ion and electron scales.
  1. Simulations:
     - Gyrokinetic
       Capture Alfvenic fluctuations, however it operates at low frequency limit and miss high k physics (dissipation scale intermittency, whistler, magnetosonic waves). It also averages out cyclotron motions.
     - Hybrid
       Capture ion kinetics. However, it misses electron kinetics.
     - Fully electromagnetic particle-in-cell (PIC) simulations
       Self-consistent, solve both ion and electron kinetics, computationally expensive
  2. Observations: high cadent spacecraft data
PIC simulation: Spectrum resolved to electron scales

Wu et al., 2013, APJL
The eddies interact **nonlinearly**, merge, stretch, attract, and repel each other, similar to a previous MHD simulation by Matthaeus and Montgomery (1980) and Servidio et al. (2009, PRL)

Movie made from a simulation in Wu et al., 2013, APJL
Reconnection X-points

Hint: intermittency

Simulation from Wu et al., 2013, APJL
Intermittency

Bruno et al., 2001
Local variability: under the same solar wind conditions, there is a broad range of local cascade rates that deviates from Gaussian. (Coburn et al. 2014)
Intermittency: PIC simulations and observations

PIC simulations. PDF(δb(l)) deviates more from Gaussian as scale l (in the figure donated by δr) is reduced.

Wu et al., 2013, APJL
Coherent structures and wave Excitation (VPIC simulation)

Heating at coherent structures (current sheet) is orders of magnitude more efficient than wave damping!

Karimabadi et al., 2013, PoP
Enhanced dissipation @ enhanced “filaments” (stronger current sheet)

Intermittent dissipation!

\[ PVI_l = \frac{|\delta b(l)|}{\sigma^2 \delta b(l)} \]

Wu et al., 2013, APJL
von Karman energy decay in MHD

Politano and Pouquet, 1998, PRE and Wan et al., 2012, JFM

Write Elsasser energies \( Z_\pm^2 = |z_\pm|^2 = |u \pm b|^2 \), here \( Z_\pm \) is the turbulent amplitude. Derive

\[
\frac{dZ_+^2}{dt} = -\alpha_+ \frac{Z_+^2 Z_-}{L_+} \quad \text{and} \quad \frac{dZ_-^2}{dt} = -\alpha_- \frac{Z_-^2 Z_+}{L_-}
\]

\[
\frac{dL_+}{dt} = \beta_+ Z_- \quad \text{and} \quad \frac{dL_-}{dt} = \beta_- Z_+
\]

Energy containing eddies \((Z_+, Z_-, L_+, L_-)\) controls decay (dissipation) \( \varepsilon = \frac{d(Z_+^2 + Z_-^2)}{dt} \), independent of viscosity and resistivity (details of dissipation mechanism)!
von Karman similarity decay in fully electromagnetic particle-in-cell simulations

Plasma energy decay appears to be consistent with MHD extension of von Karman similarity decay, independent of microphysics!

Wu et al., 2013, PRL
Remarks

Kinetic scale intermittency not only shares basic properties with its MHD and hydrodynamic counterparts, but also admits interesting differences associated with plasma effects. The coexistence of dissipative coherent structure and incoherent plasma waves makes the study of turbulence in a plasma more challenging than in ordinary fluid.